

On a Class of Riemann-Cartan Space-times of Gödel Type

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Abstract

A class of Riemann-Cartan Gödel-type space-times is examined by using the equivalence problem techniques, as formulated by Fonseca-Neto *et al.* and embodied in a suite of computer algebra programs called TCLASSI. A coordinate-invariant description of the gravitational field for this class of space-times is presented. It is also shown that these space-times can admit a group G_r of affine-isometric motions of dimensions $r = 2, 4, 5$. The necessary and sufficient conditions for space-time (ST) homogeneity of this class of space-times are derived, extending previous works on Gödel-type space-times. The equivalence of space-times in the ST homogeneous subclass is studied, recovering recent results under different premises. The results of the limiting Riemannian case are also recovered.

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1 Introduction

The Gödel solution of Einstein's field equations [1] is a particular case of the Gödel-type line element

$$ds^2 = [dt + H(x) dy]^2 - D^2(x) dy^2 - dx^2 - dz^2, \quad (1.1)$$

in which

$$H(x) = e^{mx}, \quad D(x) = e^{mx}/\sqrt{2}, \quad (1.2)$$

where m is a real constant, which is related to the cosmological constant Λ and the rotation ω by $m^2 = 2\omega^2 = -2\Lambda$. The Gödel model is homogeneous in space-time (ST homogeneous), since it admits a five-dimensional isometry group G_5 , with an isotropy subgroup of dimension one (H_1).

Despite its striking properties, the cosmological solution presented by Gödel has a well recognized historical (and even philosophical [2]) importance and has to a large extent motivated the investigation on rotating cosmological space-times. Particularly, the search for rotating Gödel-type space-times has received a great deal of attention in recent years, and the literature on these geometries is fairly large today (see, for example, Krasiński [3], Singh and Agrawal [4], and references therein).

In general relativity (GR), the space-time M is a four-dimensional *Riemannian* manifold M endowed with a locally Lorentzian metric g_{ab} . In GR there is a unique metric-compatible symmetric connection $\{^a_{bc}\}$ (Christoffel's symbols). However, it is well known that the metric tensor and the connection can be introduced as independent structures on a given space-time manifold M . In the framework of torsion theories of gravitation (TTG), we have *Riemann-Cartan* (RC) manifolds, i.e., space-time manifolds endowed with locally Lorentzian metrics g_{ab} and metric-compatible nonsymmetric connections Γ^a_{bc} . Thus, in TTG the connection has a metric-independent part given by the torsion, and for a characterization of the local gravitational field one has to deal with both metric and connection.

In GR and TTG the arbitrariness in the choice of coordinates gives rise to the problem of deciding whether or not two apparently different space-time solutions of the field equations are locally the same (the equivalence problem). In GR this problem can be couched in terms of local isometry, whereas in TTG besides local isometry ($g_{ab} \rightarrow \tilde{g}_{ab}$) it means local affine collineation ($\Gamma^a_{bc} \rightarrow \tilde{\Gamma}^a_{bc}$) of two RC manifolds.

The local equivalence for the Riemannian space-times of general relativity has been discussed by several authors and is of interest in many contexts [5] – [7]. The conditions for the local equivalence of Riemann-Cartan space-times in the torsion theories of gravitation, however, have been found only recently [8]. Subsequently, an algorithm for checking the equivalence in TTG and a working version of a computer algebra package (called TCLASSI) which implements this algorithm have been presented [9] – [11].

The problem of space-time homogeneity (ST homogeneity) of four-dimensional *Riemannian* manifolds endowed with a Gödel-type metric (1.1) was considered for the first time by Raychaudhuri and Thakurta [12] in 1980. They have shown that the conditions

$$\frac{H'}{D} = \text{const} \equiv 2\omega, \quad (1.3)$$

$$\frac{D''}{D} = \text{const} \equiv m^2 \quad (1.4)$$

are necessary for a Riemannian Gödel-type space-time manifold to be ST-homogeneous. Here and in what follows we use the prime to denote derivative with respect to x . In 1983, it was proved [13] that the conditions (1.3) and (1.4) are also sufficient for ST homogeneity of Riemannian Gödel-type space-time manifolds.

However, in both articles [12, 13] in the study of ST homogeneity it was explicitly or implicitly assumed that Gödel-type space-time manifolds can admit only time-independent Killing vector fields [14]. The conditions (1.3) and (1.4) were finally proved to be the necessary and sufficient conditions for a Riemannian Gödel-type space-time manifold to be ST homogeneous in a more general setting in [15], where the powerful equivalence problem techniques for Riemannian space-times, as formulated by Karlhede [6] and implemented in the computer algebra package CLASSI [16] were used.

In a recent work Åman *et al.* [17] have used the equivalence problem techniques for TTG to study the Riemann-Cartan manifolds endowed with a Gödel-type metric (1.1) and a torsion $T^t_{xy} = D(x)S(x)$ in the same coordinate system relative to which the metric (1.1) is given. Since in the context of Einstein-Cartan theory a torsion with only this nonvanishing component corresponds to a Weyssenhoff fluid whose vector associated to the spin density is aligned along the direction of the rotation vector (the z axis) [18, 19], through out this article we shall refer to this torsion as polarized (aligned) along the rotation vector. Clearly this torsion also shares the same translational symmetries of the metric (1.1).

In this work, in the light of the equivalence techniques, as formulated by Fonseca-Neto *et al.* [8] and embodied in the suite of computer algebra programs TCLASSI [9] – [11], we extend the above-mentioned investigations by examining a class of Riemann-Cartan Gödel-type space-times in which the torsion, although polarized along the direction of the rotation, does not share the same translational symmetries of the metric (1.1). A coordinate-invariant description of the gravitational field for this class of RC space-times is presented. We show that these RC space-times admit a group G_r of affine-isometric motions with dimensions $r = 2$ [when $H(x)$, $D(x)$ and $S(z)$ are arbitrary smooth real functions], $r = 4$ (when the conditions for ST homogeneity of the corresponding Riemannian case hold), and $r = 5$ [when besides the conditions (1.3) – (1.4) one has a constant torsion]. We also show that the RC Gödel-type space-times that allow a G_4 of symmetry are not ST homogeneous. Actually the orbit of an arbitrary point P under the action of G_4 in this class of manifolds is a three-dimensional hypersurface, and G_4 admits a subgroup of isotropy H_1 . It emerges from our results that the necessary and sufficient conditions for space-time ST homogeneity found in [17] also hold for the class of Riemann-Cartan Gödel-type space-times in which the torsion does not share the same translational symmetries of the metric.

Our major aim in the next section is to present a summary of some important prerequisites for Section 3, to set our framework, define the notation, and make our text to

a certain extent clear and self-contained.

In Section 3 we present our main results and conclusions in connection with earlier results on Gödel-type space-time manifolds [12, 13, 15, 17].

2 Theoretical and Practical Preliminaries

A solution to the equivalence problem for Riemann-Cartan manifolds can be summarized as follows [8, 9]. Two n -dimensional Riemann-Cartan manifolds M and \widetilde{M} are locally equivalent if there exist coordinate and Lorentz transformations such that the following equations relating the Lorentz frame components of the curvature and torsion tensors and their covariant derivatives:

$$\begin{aligned}
T^A_{BC} &= \tilde{T}^A_{BC}, \\
R^A_{BCD} &= \tilde{R}^A_{BCD}, \\
T^A_{BC;M_1} &= \tilde{T}^A_{BC;M_1}, \\
R^A_{BCD;M_1} &= \tilde{R}^A_{BCD;M_1}, \\
T^A_{BC;M_1M_2} &= \tilde{T}^A_{BC;M_1M_2}, \\
&\vdots \\
R^A_{BCD;M_1\dots M_{p+1}} &= \tilde{R}^A_{BCD;M_1\dots M_{p+1}}, \\
T^A_{BC;M_1\dots M_{p+2}} &= \tilde{T}^A_{BC;M_1\dots M_{p+2}}
\end{aligned} \tag{2.1}$$

are compatible as *algebraic* equations in (x^a, ξ^A) and $(\tilde{x}^a, \tilde{\xi}^A)$. Here and in what follows we use a semicolon to denote covariant derivatives. Note that x^a and \tilde{x}^a are coordinates on the manifolds M and \widetilde{M} , respectively, while ξ^A and $\tilde{\xi}^A$ parametrize the corresponding groups of allowed frame transformations. Reciprocally, equations (2.1) imply local equivalence between the space-time manifolds.

In practice, the coordinates and Lorentz transformations parameters are treated differently. Actually a fixed frame is chosen to perform the calculations so that only coordinates appear in the components of the curvature and the torsion tensors; there is no explicit

dependence on the Lorentz parameters.

It is worth noting that in calculating the covariant derivatives of the curvature and torsion tensors one can stop as soon as one reaches a step of the differentiation process at which the p^{th} derivatives (say) are algebraically expressible in terms of the previous ones, since further differentiation will not yield any new piece of information. Actually, if the p^{th} derivative is expressible in terms of its predecessors, for any $q > p$ the q^{th} derivatives can all be expressed in terms of the $0^{th}, 1^{st}, \dots, (p-1)^{th}$ derivatives. As in the worst case we have only one functionally independent function in each step of the differentiation process, it follows that for four-dimensional Riemann-Cartan manifolds $p+1 \leq 10$.

An important practical point to be considered, once one wishes to test the local equivalence of two Riemann-Cartan manifolds, is that before attempting to solve eqs. (2.1) one can extract and compare partial pieces of information as, for example, the subgroup H_q of the symmetry group G_r under which the set

$$I_q = \{T_{BC}^A, R_{BCD}^A, T_{BC;M_1}^A, R_{BCD;M_1}^A, T_{BC;M_1 M_2}^A, \dots, R_{BCD;M_1 \dots M_q}^A, T_{BC;M_1 \dots M_{q+1}}^A\}$$

is invariant, and the number t_q of functionally independent functions of the space-time coordinates contained in I_q . They must be the same at each step q ($0 \leq q \leq p+1$) of the differentiation process if the Riemann-Cartan manifolds are locally equivalent.

A practical procedure for testing equivalence of Riemann-Cartan space-times, which results from the above considerations, starts by setting $q = 0$ and has the following steps [9, 10, 20]:

1. Calculate the set I_q [the derivatives of the curvature up to the q^{th} order and of the torsion up to the $(q+1)^{th}$ order].
2. Fix the frame, as much as possible, by putting the elements of I_q into canonical forms, and find the residual isotropy group H_q of transformations which leave these canonical forms invariant.
3. Find the number t_q of functionally independent functions of space-time coordinates

in the elements of I_q , brought into the canonical forms.

4. If the isotropy groups H_q and $H_{(q-1)}$ are the same, and the number of functionally independent functions t_q is equal to $t_{(q-1)}$, then let $q = p + 1$ and stop. Otherwise, increment q by 1 and go to step 1.

To compare two Riemann-Cartan space-times we first test if they have the same t_q and H_q for each q up to $p + 1$ [$(p + 2)^{th}$ derivative of the torsion]. If they differ, so do (locally) the Riemann-Cartan manifolds. If not, it is necessary to check the consistency of eqs. (2.1).

Since there are t_p essential space-time coordinates when the above procedure for testing equivalence terminates, clearly $4 - t_p$ are ignorable, so the isotropy group will have dimension $s = \dim(H_p)$, and the group of symmetries (called affine isometries) of both metric (isometry) and torsion (affine collineations) will have dimension r given by (see, for example, refs. [5] – [10])

$$r = s + 4 - t_p, \quad (2.2)$$

acting on an orbit with dimension

$$d = r - s = 4 - t_p. \quad (2.3)$$

In our implementation of the above practical procedure, rather than using the curvature and torsion tensors as such, the algorithms and computer algebra programs were devised and written in terms of spinor equivalents, namely [9, 10]: (i) the irreducible parts of the Riemann-Cartan curvature, i.e., Ψ_{ABCD} , $\Phi_{ABX'Z'}$, $\Theta_{ABX'Z'}$, Σ_{AB} , Λ and Ω ; and (ii) the irreducible parts of torsion, i.e., $\mathcal{T}_{AX'}$, $\mathcal{P}_{AX'}$ and $\mathcal{L}_{ABCX'}$.

A relevant point to be taken into account when one needs to compute derivatives of the curvature and the torsion tensors is that they are interrelated by the Bianchi and Ricci identities and their concomitants. Thus, to cut down the number of quantities to be calculated it is very important to find a set of quantities from which the curvature and torsion tensors, and their covariant derivatives are obtainable by algebraic operations.

For Riemann-Cartan space-time manifolds, instead of using I_{p+1} as such, we deal with a corresponding complete minimal set of quantities which are recursively defined in terms of totally symmetrized q^{th} and $(q+1)^{th}$ (for $0 \leq q \leq p+1$) derivatives of the curvature and torsion spinors, respectively [9, 10, 21]. In this work, however, we shall only need the subsets of quantities for $q = 0$ and $q = 1$, which can be taken to be (see [9, 10]) the quantities tabulated, respectively, in the tables 1 and 2 below, where, to give an idea of the amount of calculations involved in the equivalence procedure, we have also included the number of real independent components of each spinorial quantities in the general (worst) case.

TCLASSI's name	Spinor	Ind. Comp.
TPSI	Ψ_{ABCD}	10
PSILTOR	ψ_{ABCD}	10
TPHI	$\Phi_{ABX'Y'}$	9
PHILTOR	$\phi_{ABX'Y'}$	9
THETA	$\Theta_{ABX'Y'}$	9
DSPTTOR	$\nabla^{(B}_{(B'}\mathcal{T}_{X')}^A)$	9
DSPPTOR	$\nabla^{(B}_{(B'}\mathcal{P}_{X')}^A)$	9
SIGMA	Σ_{AB}	6
BVTTOR	\mathcal{M}_{AB}	6
BVPTOR	\mathcal{B}_{AB}	6
SPTTOR	$\mathcal{T}_{AX'}$	4
SPPTOR	$\mathcal{P}_{AX'}$	4
TLAMBD	Λ	1
OMEGA	Ω	1
SCTTOR	\mathcal{T}	1
SPLTTOR	$\mathcal{L}_{ABCX'}$	16
DSPLTOR	$\nabla^{(B}_{(B'}\mathcal{L}_{X')}^{CDE)}$	30

Table 1: The 17 quantities from a complete minimal set for the step $q = 0$ of the equivalence algorithm ($I_0 = \{T_{BC}^A, R_{BCD}^A, T_{BC;M_1}^A\}$). TCLASSI's names and the number of real independent components are also shown. In the general case there are a total of 140 real components.

TCLASSI's name	Spinor	Ind. Comp.
DTPSI	$\nabla_{X'(E)} \Psi_{ABCD}$	24
DPSILTOR	$\nabla_{X'(E)} \psi_{ABCD}$	24
DTPHI	$\nabla^{(C}_{(X'} \Phi^{AB)}_{Y'Z')}$	16
DPHILTOR	$\nabla^{(C}_{(X'} \phi^{AB)}_{Y'Z')}$	16
DTHETA	$\nabla^{(C}_{(X'} \Theta^{AB)}_{Y'Z')}$	16
D2SPTTOR	$\nabla^{(C}_{(X'} \nabla^A_{Y'} \mathcal{T}^B_{Z')}$	16
D2SPPTOR	$\nabla^{(C}_{(X'} \nabla^A_{Y'} \mathcal{P}^B_{Z')}$	16
DSIGMA	$\nabla_{X'(A)} \Sigma_{BC}$	16
DBVTTOR	$\nabla_{X'(A)} \mathcal{M}_{BC}$	16
DBVPTOR	$\nabla_{X'(A)} \mathcal{B}_{BC}$	16
ASPTTOR	$\square \mathcal{T}_{AX'}$	4
ASPPTOR	$\square \mathcal{P}_{AX'}$	4
DTLAMBD	$\nabla_{X'A} \Lambda$	4
DOMEGA	$\nabla_{X'A} \Omega$	4
DSCTTOR	$\nabla_{X'A} \mathcal{T}$	4
D2SPLTOR	$\nabla^{(A}_{(X'} \nabla^B_{Y'} \mathcal{L}^{CDE)}_{Z')}$	48
ASPLTOR	$\square \mathcal{L}_{X'ABC}$	16
TXI	$\Xi_{X'ABC}$	16
XITH	$\mathcal{X}_{X'ABC}$	16
TSIGM	$\mathcal{U}_{AX'}$	4
PSIGM	$\mathcal{V}_{AX'}$	4

Table 2: The 21 quantities which, together with the 17 quantities of table 1, form a complete minimal set for the step $q = 1$ of the equivalence algorithm ($I_1 = I_0 \cup \{R^A_{BCD;M_1}, T^A_{BC;M_1M_2}\}$). TCLASSI's names and the number of real independent components are also shown. These 21 quantities have a total of 300 real components in the general case.

To close this section we remark that, in line with the usage in the literature, in the TCLASSI implementation of the above results it is used a notation in which the indices are all subscripts, components are labelled by a primed and unprimed index whose numerical values are the sum of corresponding (prime and unprimed) spinors indices. Thus, for example, one has $\nabla \Psi_{20'} \equiv \Psi_{(1000;1)0'}$, where the parentheses indicate symmetrization.

3 Main Results and Conclusions

The basic idea behind our procedure for checking the local equivalence of RC space-times, discussed in the previous section, is a separate handling of frame rotations and space-time coordinates, fixing the frame at each stage of differentiation (of the curvature and torsion tensors) by aligning the basis vectors as far as possible with invariantly-defined directions. This is done in practice, by bringing to canonical forms first the quantities with the same symmetry as the Weyl spinor (the Weyl-type spinors: Ψ_A and ψ_A) followed by the spinors with the symmetry of the Ricci spinor (the Ricci-type spinors: $\Phi_{AB'}$, $\phi_{AB'}$, $\Theta_{AB'}$, $\nabla \mathcal{T}_{AX'}$, $\nabla \mathcal{P}_{AY'}$), then bivector spinors (Σ_{AB} , \mathcal{M}_{AB} , \mathcal{B}_{AB}), and finally vectors ($\mathcal{T}_{AX'}$, $\mathcal{P}_{AX'}$) are taken into account. Thus, if Ψ_A is Petrov type D, for example, the frame is fixed by demanding that the only nonvanishing component of Ψ_A is Ψ_2 . On the other hand, if Ψ_A is Petrov type I the frame can be fixed by requiring that the components of Ψ_A are such that $\Psi_1 = \Psi_3 \neq 0$, $\Psi_2 \neq 0$. Clearly an alternative canonical frame for Petrov type I is obtained by imposing $\Psi_0 = \Psi_4 \neq 0$, $\Psi_2 \neq 0$. Although the latter choice is implemented in TCLASSI as the canonical frame for Petrov type I, in this section we shall use the former (defined to be an acceptable alternative in TCLASSI) to make straightforward the comparison of our findings with the previous results on Gödel-type space-times [15, 17].

We shall consider now a class of four-dimensional Riemann-Cartan manifolds M , endowed with a Gödel-type metric (1.1) and a torsion that is aligned with the preferred direction defined by the rotation vector field, but which does not share the same trans-

lational symmetries of the metric (1.1). Actually the class of RC space-times we are concerned with here is such that in the coordinate system in which (1.1) is given, the nonvanishing components of the torsion reduce to

$$T^t_{xy} \equiv D(x) S(z). \quad (3.1)$$

It should be emphasized that in the Lorentz frame relative to which the Gödel-type line element (1.1) reduces to

$$ds^2 = \eta_{AB} \omega^A \omega^B \quad \text{with} \quad \eta_{AB} = \text{diag}(+1, -1, -1, -1), \quad (3.2)$$

and

$$\omega^0 = dt + H(x) dy, \quad \omega^1 = dx, \quad \omega^2 = D(x) dy, \quad \omega^3 = dz, \quad (3.3)$$

the only nonvanishing component of the torsion is

$$T^0_{12} = S(z). \quad (3.4)$$

Therefore, the function $D(x)$ in (3.1) and in expression for the torsion in Åman *et al.* [17] can be eliminated by a suitable choice of basis.

For arbitrary functions $H(x)$, $D(x)$ and $S(z)$, the Weyl-type spinor Ψ_A is Petrov type I, whereas ψ_A is Petrov type D; this fact can be easily checked by using the module SEGPET of TCLASSI. Accordingly the null tetrad θ^A which turns out to be appropriate (canonical) for our purpose here is

$$\begin{aligned} \theta^0 &= \frac{1}{\sqrt{2}} [dt + H(x) dy + dz], & \theta^2 &= \frac{1}{\sqrt{2}} [D(x) dy - i dx], \\ \theta^1 &= \frac{1}{\sqrt{2}} [dt + H(x) dy - dz], & \theta^3 &= \frac{1}{\sqrt{2}} [D(x) dy + i dx]. \end{aligned} \quad (3.5)$$

Clearly in this basis the Gödel-type line element (1.1) and the torsion tensor T are, respectively, given by

$$ds^2 = 2(\theta^0 \theta^1 - \theta^2 \theta^3) \quad \text{and} \quad T^0_{23} = T^1_{23} = \frac{\sqrt{2}}{2} i S(z). \quad (3.6)$$

It is worth mentioning that the Petrov type for Ψ_A and ψ_A and the canonical frame (3.5) were obtained by interaction with TCLASSI, starting from a Lorentz frame, changing to a null tetrad frame, and making dyad transformations to bring Ψ_A and ψ_A , respectively, into the canonical form for Petrov types I and D. As a matter of fact, in the frame (3.5) all quantities of the sets I_0 and I_1 are brought into their corresponding canonical forms.

Using the TCLASSI package one finds the following nonvanishing components of the quantities in the complete minimal set corresponding to I_0 (see table 1) of our algorithm:

$$\Psi_1 = \Psi_3 = -\frac{1}{8} \left(\frac{H'}{D} \right)', \quad (3.7)$$

$$\Psi_2 = -\frac{S}{4} \left(\frac{S}{3} - \frac{H'}{D} \right) + \frac{1}{6} \left[\frac{D''}{D} - \left(\frac{H'}{D} \right)^2 \right] + \frac{i}{6} \dot{S}, \quad (3.8)$$

$$\psi_2 = -\frac{S}{4} \left(S - \frac{H'}{D} \right) + \frac{i}{6} \dot{S}, \quad (3.9)$$

$$\Phi_{00'} = \Phi_{22'} = \frac{S}{4} \left(\frac{S}{2} - \frac{H'}{D} \right) + \frac{1}{8} \left(\frac{H'}{D} \right)^2, \quad (3.10)$$

$$\Phi_{01'} = \Phi_{12'} = \frac{1}{8} \left(\frac{H'}{D} \right)', \quad (3.11)$$

$$\Phi_{11'} = \frac{S}{4} \left(\frac{S}{4} - \frac{H'}{D} \right) + \frac{1}{4} \left[\frac{3}{4} \left(\frac{H'}{D} \right)^2 - \frac{D''}{D} \right], \quad (3.12)$$

$$\phi_{00'} = \phi_{22'} = \phi_{11'} = \frac{S}{4} \left(S - \frac{H'}{D} \right), \quad (3.13)$$

$$\Theta_{00'} = \Theta_{22'} = 2\Theta_{11'} = \frac{\dot{S}}{4}, \quad (3.14)$$

$$\nabla \mathcal{P}_{00'} = \nabla \mathcal{P}_{22'} = -2\nabla \mathcal{P}_{11'} = -\frac{\dot{S}}{2}, \quad (3.15)$$

$$\mathcal{P}_{00'} = -\mathcal{P}_{11'} = -\frac{\sqrt{2}}{2} S, \quad (3.16)$$

$$\Lambda = -\frac{S^2}{48} - \frac{1}{12} \left[\frac{D''}{D} - \frac{1}{4} \left(\frac{H'}{D} \right)^2 \right], \quad (3.17)$$

$$\Omega = \frac{\dot{S}}{24}, \quad (3.18)$$

$$\mathcal{L}_{10'} = \mathcal{L}_{21'} = -\frac{i}{6}\sqrt{2} S, \quad (3.19)$$

$$\nabla\mathcal{L}_{10'} = -\nabla\mathcal{L}_{32'} = \frac{S}{16}\left(S - \frac{H'}{D}\right) - \frac{i}{8}\dot{S}, \quad (3.20)$$

where the prime and overdot denote, respectively, derivative with respect to x and z . Note, incidentally, that ψ_A is Petrov type D for this class while for the RC Gödel-type space-times discussed in [11] both Weyl-type spinor Ψ_A and ψ_A are Petrov type I, making clear that the underlying RC manifolds are not (locally) equivalent, as one may have noticed from the outset.

Before proceeding to the second step of our practical procedure let us introduce a notation. In line with the usage in the literature (see, for example, [23]) we shall refer to the spinorial (frame) components of the quantities in the sets I_q (and the quantities of the corresponding complete minimal sets) for each step q of the equivalence problem algorithm as *Cartan scalars*, since they are scalars under coordinate transformations.

Following the algorithm of the previous section, one needs to find the isotropy group which leaves the above Cartan scalars (canonical forms) invariant, and the number of functionally independent functions of the space-time coordinates among these Cartan scalars. One can easily find that the above whole set of Cartan scalars (3.7) – (3.20) is not invariant under any subgroup of the Lorentz group, so $\dim(H_0) = 0$. Moreover, for arbitrary smooth real functions $H(x)$, $D(x)$ and $S(z)$ the number of functionally independent functions of the coordinates in the complete minimal set corresponding to I_0 clearly is $t_0 = 2$.

The next steps of our algorithm are (i) to calculate the Cartan scalars of table 2, (ii) to find the residual isotropy group which leaves these quantities invariant, and (iii) to find if there is any additional functionally independent function of the coordinates in the set of Cartan scalars of table 2. For the sake of brevity we shall present the results without going into details of the calculations, which can be easily worked out by using TCLASSI. The nonvanishing Cartan scalars of table 2 are DTPSI, DPSILTOR, DTPHI, DPHILTOR, DTHETA, D2SPLTOR, D2SPPTOR, ASPLTOR, ASPPTOR, DTLAMBD, DOMEGA, TXI, and XITH. These

scalars are invariant under no subgroup of the Lorentz group [$\dim(H_0) = \dim(H_1) = 0$], and are such that $t_1 = t_0 = 2$. Thus, no new covariant derivative should be calculated. From eq. (2.2) one finds that the group of symmetries (affine-isometric motions) of this class of Riemann-Cartan Gödel-type space-times is two-dimensional.

We shall now consider the class of Riemann-Cartan Gödel-type space-time manifolds where the underlying *Riemannian* manifolds are ST homogeneous [13, 15], i.e., when the conditions (1.3) and (1.4) hold. In this case the nonvanishing Cartan scalars corresponding to the first step of our algorithm for $q = 0$ reduce to

$$\Psi_2 = \frac{S}{2} \left(\omega - \frac{S}{6} \right) + \frac{m^2}{6} - \frac{2}{3} \omega^2 + \frac{i}{6} \dot{S}, \quad (3.21)$$

$$\psi_2 = -\frac{S}{4} (S - 2\omega) + \frac{i}{6} \dot{S}, \quad (3.22)$$

$$\Phi_{00'} = \Phi_{22'} = \frac{S}{4} \left(\frac{S}{2} - 2\omega \right) + \frac{\omega^2}{2}, \quad (3.23)$$

$$\Phi_{11'} = \frac{S}{4} \left(\frac{S}{4} - 2\omega \right) + \frac{3}{4} \omega^2 - \frac{m^2}{4}, \quad (3.24)$$

$$\phi_{00'} = \phi_{22'} = \phi_{11'} = \frac{S}{4} (S - 2\omega), \quad (3.25)$$

$$\Theta_{00'} = \Theta_{22'} = 2\Theta_{11'} = \frac{\dot{S}}{4}, \quad (3.26)$$

$$\nabla \mathcal{P}_{00'} = \nabla \mathcal{P}_{22'} = -2\nabla \mathcal{P}_{11'} = -\frac{\dot{S}}{2}, \quad (3.27)$$

$$\mathcal{P}_{00'} = -\mathcal{P}_{11'} = -\frac{\sqrt{2}}{2} S, \quad (3.28)$$

$$\Lambda = -\frac{S^2}{48} + \frac{1}{12} (\omega^2 - m^2), \quad (3.29)$$

$$\Omega = \frac{\dot{S}}{24}, \quad (3.30)$$

$$\mathcal{L}_{10'} = \mathcal{L}_{21'} = -\frac{i}{6} \sqrt{2} S, \quad (3.31)$$

$$\nabla \mathcal{L}_{10'} = -\nabla \mathcal{L}_{32'} = \frac{S}{16} (S - 2\omega) - \frac{i}{8} \dot{S}. \quad (3.32)$$

Following the algorithm of the previous section, one needs to find the isotropy group which leaves the above Cartan scalars (canonical forms) invariant as well as the number of functionally independent functions of the coordinates among these Cartan scalars. Clearly as S is an arbitrary smooth real function one has $t_0 = 1$. As far as the isotropy group

H_0 is concerned, since $S \neq 0$ one can easily find that although there are Cartan scalars as, e.g., Ω and Λ which are invariant under the Lorentz group, the whole set of Cartan scalars (3.21) – (3.32) is invariant only under the subgroup of spatial rotation

$$\begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix}, \quad (3.33)$$

where α is a real parameter. So, the isotropy group H_0 is such that $\dim H_0 = 1$.

For the following step ($q = 1$) of our algorithm one readily finds

$$\begin{aligned} \nabla \Psi_{20'} &= -\nabla \Psi_{31'} = \frac{i}{40} \sqrt{2} S (2m^2 - 20\omega^2 + 8\omega S - S^2) \\ &\quad + \frac{i}{10} \sqrt{2} \omega (4\omega^2 - m^2) + \frac{\sqrt{2}}{20} [(5\omega - 2S)\dot{S} + i\ddot{S}], \end{aligned} \quad (3.34)$$

$$\begin{aligned} \nabla \psi_{20'} &= -\nabla \psi_{31'} = -\frac{i}{40} 3\sqrt{2} S (4\omega^2 - 4\omega S + S^2) \\ &\quad + \frac{\sqrt{2}}{20} [(5\omega - 4S)\dot{S} + i\ddot{S}], \end{aligned} \quad (3.35)$$

$$\nabla \Phi_{00'} = -\nabla \Phi_{33'} = \frac{\sqrt{2}}{8} (S - 2\omega) \dot{S}, \quad (3.36)$$

$$\nabla \Phi_{11'} = -\nabla \Phi_{22'} = \frac{\sqrt{2}}{72} (S - 6\omega) \dot{S}, \quad (3.37)$$

$$\nabla \phi_{00'} = -\nabla \phi_{33'} = 3\nabla \phi_{11'} = -3\nabla \phi_{22'} = \frac{\sqrt{2}}{4} (S - \omega) \dot{S}, \quad (3.38)$$

$$\nabla \Theta_{00'} = -\nabla \Theta_{33'} = 9\nabla \Theta_{11'} = -9\nabla \Theta_{22'} = \frac{\sqrt{2}}{8} \ddot{S}, \quad (3.39)$$

$$\nabla^2 \mathcal{P}_{00'} = -\nabla^2 \mathcal{P}_{33'} = -3\nabla^2 \mathcal{P}_{11'} = 3\nabla^2 \mathcal{P}_{22'} = -\frac{\sqrt{2}}{4} \ddot{S}, \quad (3.40)$$

$$\square \mathcal{P}_{00'} = -\square \mathcal{P}_{11'} = \frac{\sqrt{2}}{2} \ddot{S}, \quad (3.41)$$

$$\nabla \Lambda_{00'} = -\nabla \Lambda_{11'} = -\frac{\sqrt{2}}{48} S \dot{S}, \quad (3.42)$$

$$\nabla \Omega_{00'} = -\nabla \Omega_{11'} = \frac{\sqrt{2}}{48} \ddot{S}, \quad (3.43)$$

$$\begin{aligned} \nabla^2 \mathcal{L}_{10'} &= \nabla^2 \mathcal{L}_{43'} = \frac{i}{80} \sqrt{2} S (4\omega^2 - 4S\omega + S^2) \\ &\quad + \frac{1}{40} \sqrt{2} [(3S - 4\omega)\dot{S} - 2i\ddot{S}], \end{aligned} \quad (3.44)$$

$$\nabla^2 \mathcal{L}_{21'} = \nabla^2 \mathcal{L}_{32'} = -\frac{i}{480} \sqrt{2} S (4\omega^2 - 4S\omega + S^2)$$

$$+\frac{1}{240}\sqrt{2}\left[(4\omega-3S)\dot{S}+2i\ddot{S}\right], \quad (3.45)$$

$$\square\mathcal{L}_{10'} = \square\mathcal{L}_{21'} = \frac{i}{4}\sqrt{2}S(4\omega^2-4\omega S+S^2) + \frac{i}{6}\sqrt{2}\ddot{S}, \quad (3.46)$$

$$\begin{aligned} \Xi_{10'} = \Xi_{21'} = & \frac{i}{16}\sqrt{2}S(2m^2+8S\omega-S^2-20\omega^2) \\ & + \frac{i}{4}\sqrt{2}\omega(4\omega^2-m^2) - \frac{\sqrt{2}}{24}(S\dot{S}+2i\ddot{S}), \end{aligned} \quad (3.47)$$

$$\mathcal{X}_{10'} = \mathcal{X}_{21'} = -\frac{\sqrt{2}}{24}\left[(3S-6\omega)i\dot{S}+2\ddot{S}\right]. \quad (3.48)$$

As no new functionally independent function came out, then $t_0 = t_1 = 1$. Besides, the Cartan scalars (3.34) – (3.48) are invariant under the same isotropy group (3.33), i.e. $H_0 = H_1$. Thus no new covariant derivative should be calculated. From eq. (2.2) one finds that the group of symmetries (affine-isometric motions) of this particular class of Riemann-Cartan Gödel-type space-times is four-dimensional. Note, however, that these RC manifolds are not ST homogeneous (see below for a formal definition of ST homogeneity) as one might think at a first sight. Indeed, from eq. (2.3) one readily find that the orbit of an arbitrary point P on a manifold of this class, under the action of the group of symmetries, is three-dimensional.

We shall finally focus our attention in the ST homogeneous RC Gödel-type space-times. A word of clarification in order here: a n -dimensional Riemann-Cartan manifold M is said to be homogeneous when the orbit of an arbitrary point $P \in M$ under the action of the group of affine-isometric motions G_r is the manifold M itself. Clearly for ST homogeneity of a four-dimensional RC manifold we have an orbit of dimension $d = 4$. Now, from equation (2.3) one finds that for ST homogeneity ($d = 4$) we must have $t_q = 0$, for all steps q in the equivalence procedure. Thus, from eqs. (3.7) – (3.20) one easily concludes that for the class of Riemann-Cartan Gödel-type space-times (3.6) [or equivalently given by a metric (3.2) and (3.3), and a torsion (3.4)] to be ST homogeneous, besides the conditions (1.3) – (1.4) it is necessary that it has a constant torsion, i.e., that the condition

$$S = \text{const} \equiv S_0 \quad (3.49)$$

holds. These necessary conditions are also sufficient for ST homogeneity of our class of RC Gödel-type manifolds. Indeed, under the conditions (1.3), (1.4) and (3.49) the Cartan scalars (3.21) – (3.32) and (3.34) – (3.48) corresponding to the first and second steps of differentiation in our algorithm reduce, respectively, to eqs. (3.23) – (3.31) and eqs. (3.33) – (3.38) of [17], which imply that the corresponding RC Gödel-type space-times are ST homogeneous with a G_5 of symmetry, and characterized by three independent parameters S_0 , m^2 and ω : [identical triads (S_0, m^2, ω) specify locally equivalent RC Gödel-type space-time manifolds]. As a matter of fact, from eqs. (3.21) – (3.32) and (3.34) – (3.48), by using the equivalence problem techniques, it is straightforward to show that (1.3), (1.4) and (3.49) are the necessary and sufficient conditions for ST homogeneity of these RC Gödel-type manifolds, which admit a G_5 of symmetries.

The above results for the ST homogeneous Gödel-type class of RC space-time manifolds extend the theorems 1 and 2 of Ref. [17] (given below, for completeness) to the case in which the torsion although aligned along the direction of the rotation is does not share the same translational symmetry of the Gödel-type metric (1.1). The above-mentioned generalizations of the theorems 1 and 2 can be stated as follows:

Theorem 1 *The necessary and sufficient conditions for a Riemann-Cartan space-time manifold M endowed with a Gödel-type metric (3.2) and (3.3), and a torsion (3.4) [or equivalently with metric (1.1) and torsion (3.1)] to be ST locally homogeneous¹ are given by equations (1.3), (1.4) and (3.49).*

Theorem 2 *All ST locally homogeneous Riemann-Cartan space-time manifolds M endowed with a Gödel-type metric (3.2) and (3.3), and a torsion (3.4) [or equivalently with metric (1.1) and torsion (3.1)] admit a five-dimensional group of affine-isometric motions and are characterized by three independent parameters m^2 , ω and S_0 : identical triads (m^2, ω, S_0) specify locally equivalent manifolds.*

It should also be stressed that when $S = 0$ and the conditions (1.3) – (1.4) hold the Cartan scalars (3.21) – (3.32) and (3.34) – (3.48) reduce to the corresponding scalars

¹ For a clear and formal distinction between local and global (topological) homogeneity of a manifold see, for example, Koinke *et al.* [24].

for *Riemannian* Gödel-type space-times (eqs. (3.12) – (3.15) and (3.18) – (3.21) in [15]). Therefore, the results in [15] can be reobtained as a particular case of our study in this work.

It is worth emphasizing that by the procedure to test local equivalence we have used throughout this work, we actually compute one invariant local characterization of each class of Riemann-Cartan space-times, and at the end of the procedure in addition to t_q 's and H_q 's we have a number of consequent data as, for example, the dimension of the symmetry group [given by (2.2)], the dimension of the orbit of an arbitrary point under its action [given by (2.3)], and the algebraic classifications (Petrov and Segre) of Weyl-type and Ricci-type spinors (needed to fix the frame at the step $q = 0$ of our algorithm). Furthermore, the complete set of Cartan scalars I_{p+1} for each class or RC Gödel-type space-time manifolds give a (local) coordinate-invariant description of the gravitational field in each class of RC Gödel-type manifolds irrespective of the torsion theory of gravitation one may be concerned with (for a list of Gödel-type solutions to the Einstein-Cartan field equations, for example, see [4]). This reveals the importance of our results in the general context of the torsion theories of gravitation in which Riemann-Cartan manifolds are the underlying arena for the formulation of the physical laws.

To close this article in what follows we summarize the main results we have obtained. By using the equivalence techniques embodied in the suite of computer algebra programs (called TCLASSI) we have examined a class of Riemann-Cartan space-time manifolds M endowed with a Gödel-type metric (3.2) with (3.3), and a torsion (3.4) [or equivalently with metric (1.1) and torsion (3.1)]. We have shown that in the general case, i.e., when $H(x)$, $D(x)$ and $S(z)$ are arbitrary smooth real functions these RC space-times admit a group G_2 of affine-isometric motions. On the other hand, when the conditions for ST homogeneity (1.3) and (1.4) of the underlying Riemannian manifold are imposed the resulting family of Riemann-Cartan Gödel-type permits a group G_4 of affine-isometric motions. Moreover, when besides the conditions (1.3) – (1.4) one has a constant torsion [given by (3.4) with $S(z) = const$] the group of affine-isometric motions of these Riemann-

Cartan Gödel-type manifolds is five-dimensional. We have also derived the above theorems 1 and 2 for the case in which the torsion is polarized along the direction of the rotation but does not share the same translational symmetries of the metric (1.1), extending, therefore, the results found by Åman *et al.* [17]. Finally, the results of the Riemannian Gödel-type manifolds found by Rebouças and Åman [15] have been recovered in the limit when the torsion vanishes.

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